PROBLEM 8: COHEN-MACAULAYNESS OF ASM VARIETIES

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1. Introduction

Fulton [Ful92] introduced *matrix Schubert varieties* in 1992. They are affine varieties indexed by permutations and defined by determinantal equations one can read off the associated permutation matrix. Fulton showed that matrix Schubert varieties were Cohen–Macaulay and described the dimension of the variety in terms of Coxeter length of the associated permutation. In 2005, Knutson and Miller [KM05] used Gröbner degeneration to give another proof that all matrix Schubert varieties were Cohen–Macaulay.

Alternating sign matrices (ASMs) are a generalization of permutation matrices and have a rich history as a class of matrices. In 1983, Mills, Robbins, and Rumsey [MRR83] gave a conjecture for a closed form for the number of $n \times n$ ASMs. The original proof was given by Zeilberger [Zei96], and a second proof was given by Kuperberg [Kup96] using the six-vertex model of statistical mechanics.

The notion of an *ASM variety*, which generalizes that of a matrix Schubert variety, was introduced by Weigandt [Wei18] in 2018. ASM varieties are all of the varieties that are defined by equations enforcing northwest rank conditions in a generic matrix. They can also be described as unions of matrix Schubert varieties. Which matrix Schubert varieties appear as components of which ASM varieties is determined by a combinatorial formula shown by Weigandt in [Wei18]. Unlike matrix Schubert varieties, not all ASM varieties are Cohen–Macaulay, see Klein and Weigandt [KW21, Section 7].

The goal of this report is describe properties of Cohen–Macaulay ASM varieties and non-Cohen–Macaulay ASM varieties.

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We also investigate basic operations on ASMs, and check whether they preserve Cohen–Macaulayness. Call an ASM *Cohen–Macaulay* if its associated variety is Cohen–Macaulay. One of the main questions we investigated is related to decomposition of Cohen–Macaulay matrices.

Question 1.1. *If $A = A_1 \oplus A_2$ is an ASM, is A Cohen–Macaulay if and only if A_1 and A_2 are Cohen–Macaulay?*

The forward implication is a consequence of Lemma 4.3 and Proposition 5.1, and we have shown that for the other impliciation, it suffices to consider the simpler question:

Question 1.2. *If A is Cohen–Macaulay, is $1 \oplus A$ Cohen–Macaulay?*

We have verified using Macaulay2 that Question 1.2 has an affirmative answer for ASMs of size at most five, see Section 7. Surprisingly, the method of vertex decomposition in Knutson-Miller [KM05] used to show that matrix Schubert varieties are Cohen–Macaulay fails, as demonstrated in Section 3.

Related to Question 1.2 is whether adding a one in an arbitrary row and column preserves Cohen–Macaulayness, i.e. the following question:

Question 1.3. *If $A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)$ is an ASM and we consider*

$$\widetilde{A} = \left(\begin{array}{c|c|c} & 0 & \\ A_1 & \vdots & A_2 \\ & 0 & \\ \hline 0\dots0 & 1 & 0\dots0 \\ \hline & 0 & \\ A_3 & \vdots & A_4 \\ & 0 & \end{array}\right),$$

is $\tilde{A}$ always Cohen–Macaulay?

Although Question 1.3 does not always have an affirmative answer, as shown in Example 3.4, we have shown that adding a one in the top right corner of an ASM and in the bottom right corner of an ASM preserves Cohen–Macaulayness, see Proposition 5.2. The more broad question of adding a one in the top row remains unsolved:

Question 1.4. *Let $A = [A_1|A_2]$ be an ASM. Let*

$$\widetilde{A} = \left(\begin{array}{c|c|c} 0\dots0 & 1 & 0\dots0 \\ \hline & 0 & \\ A_1 & \vdots & A_2 \\ & 0 & \end{array}\right).$$

If A is Cohen–Macaulay, is $\tilde{A}$ Cohen–Macaulay?

In this direction, we have proved that adding a one preserves a weaker property, *height unmixedness* of the ideal I_A, see Theorem 5.3. We have also found a family of submatrices which cannot be in the top right corner of an ASM, see Proposition 4.7.

2. Background

Throughout this report, κ will denote an arbitrary field.

Definition 2.1. An *alternating sign matrix* (ASM) is square matrix with the following properties:

(1) Each entry is taken from the set $\{-1, 0, 1\}$.
(2) The entries in each row (resp. column) sum to 1.
(3) The nonzero entries in a row (resp. column) alternate between 1 and -1.

Example 2.2. *Examples of ASMs include: any permutation matrix,*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \textit{ and } \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

2.1. **ASM varieties and ideals.** We now outline the process for associating a variety and an ideal to an alternating sign matrix, following [KW21].

For each ASM $A = (A_{a,b})$, define a *rank function* rk_A on $\{1, \ldots, n\} \times \{1, \ldots, n\}$ by

$$\mathrm{rk}_A(i, j) = \sum_{a=1}^{i} \sum_{b=1}^{j} A_{a,b}.$$

The *rank matrix* of A is the $n \times n$ matrix whose (i, j) entry is $\mathrm{rk}_A(i, j)$.

Now we outline how to construct an ideal I_A from $A \in \mathtt{ASM}(n)$. First, we describe the *Rothe diagram* associated to A. For a fixed ASM A, we define functions l, u on $\{1, \ldots, n\} \times \{1, \ldots, n\}$. We set $l(i, j) = 1$ if $k := \max\{j' : 1 \le j' \le j, A_{i,j'} \neq 0\}$ exists and $A_{i,k} = 1$. Otherwise, $l(i, j) = -1$ (note that this includes the case where $\{j' : 1 \le j' \le j, A_{i,j'} \neq 0\}$ is empty and k does not exist). Similarly, we define $u(i, j) = 1$ if $k' := \max\{i' : 1 \le i' \le i, A_{i',j} \neq 0\}$ exists and $A_{k',j} = 1$; otherwise, it is -1. Then:

Definition 2.3. A *diagram box* is an ordered pair (i, j) so that both $l(i, j)$ and $u(i, j)$ are equal to -1. A visual representation of the *Rothe diagram* of A is constructed by crossing out all but the diagram boxes.

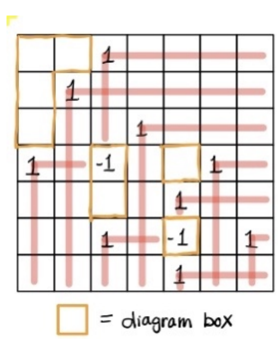


Figure 1. An ASM with Rothe diagram drawn and diagram boxes marked.

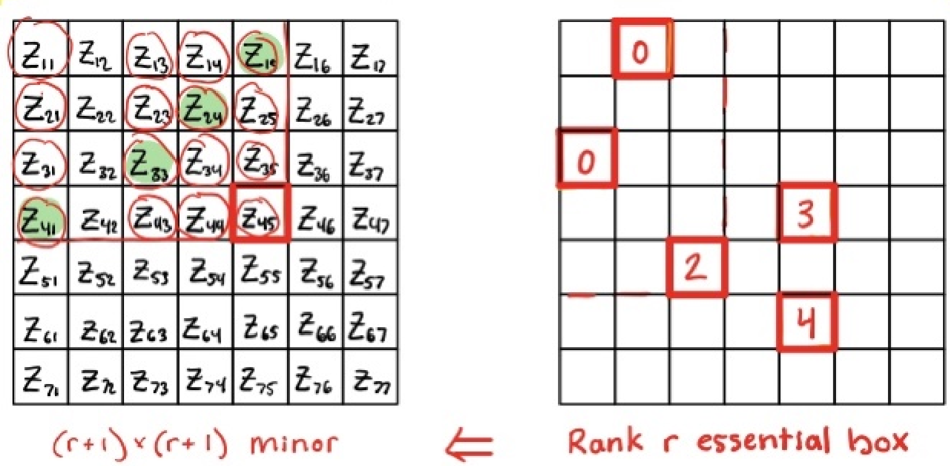


FIGURE 2. Right: The essential boxes corresponding to the matrix in Figure 1 with their rank labelled. Left: The generic matrix labelled, with one minor corresponding to the rank 3 essential box circled in red, and its anti-diagonal entries highlighted in green.

We may describe *connected components* of diagram boxes that are connected by sharing an edge (e.g. $A_{1,1}$ and $A_{2,1}$ are adjacent, but $A_{1,1}$ and $A_{2,2}$ are not). Due to the structure of an ASM, all diagram boxes in the same connected component have the same rank. We will call the set of diagram boxes with rank 0 the *dominant region* of A.

Definition 2.4. We call a diagram box $A_{i,j}$ an *essential box* if there are no other diagram boxes $A_{i',j'}$ in its connected component with $i' \geq i$ and $j' \geq j$.

Fix an $n \times n$ generic matrix $Z = (z_{ij})$ and let $R = \kappa[z_{1,1}, \ldots, z_{n,n}]$. We write $I_k(Z_{[i],[j]})$ for the ideal of R generated by the k-minors in $Z_{[i],[j]}$, where $Z_{[i],[j]}$ be the submatrix consisting of the first i rows and j columns of Z. As a convention, if $i = 0$ or $j = 0$, then define $I_k(Z_{[i],[j]}) = (0)$. The *ASM ideal* of A is

$$I_A := \sum_{(i,j) \text{ is an essential box}} I_{\mathrm{rk}_A(i,j)+1}(Z_{[i],[j]}),$$

and the generating set described here make up the *Fulton generators.* Notice that I_A is also equal to

$$\sum_{(i,j) \text{ is a diagram box}} I_{\mathrm{rk}_A(i,j)+1}(Z_{[i],[j]}).$$

We will say that the ASM A is Cohen–Macaulay whenever the ring R/I_A is Cohen–Macaulay. Also, the *ASM variety* associated to A is the variety in $\mathbb{A}^{n^2}$ defined by the ideal I_A.

We call a term order *antidiagonal* if the lead term of the determinant of a generic matrix is the product of the entries along the main anti-diagonal. We then write $\mathtt{in}\, I_A$ for the *antidiagonal initial ideal* of I_A. Then the following theorem due to Conca and Varbaro [CV20] is quite convenient:

Theorem 2.5. *If A is an ASM, then R/I_A is Cohen–Macaulay if and only if $R/\mathtt{in}\, I_A$ is Cohen–Macaulay.*

We will often be invoking this theorem without further mention.

We now review some basic definitions from simplicial complex theory.

Definition 2.6. Given a simplicial complex Δ, we define the *link* of Δ at an vertex y by $\mathtt{lk}_\Delta(y) = \{\tau \in \Delta \mid \tau \cap \{y\} = \emptyset, \tau \cup \{y\} \in \Delta\}$, and the *deletion* of Δ at a vertex y by $\mathtt{del}_\Delta(y) = \{\tau \in \Delta \mid \tau \cap \{y\} = \emptyset\}$.

We will revisit the following property of simplicial complexes in Section 3.

Definition 2.7. A simplicial complex Δ is *vertex-decomposable* if Δ is pure and either

(1) $\Delta = \{\emptyset\}$.
(2) For some vertex v in Δ, both $\mathtt{del}_\Delta(v)$ and $\mathtt{lk}_\Delta(v)$ are vertex-decomposable.

Now we discuss how these ideal from simplicial complex theory relate to the objects we want to study. The *Stanley-Reisner* correspondence constitutes a bijection between simplicial complexes on $\{1, 2, \ldots, n\}$ and squarefree monomial ideals, see [MS05, Chapter 1]. The *Stanley-Reisner* ideal of the simplicial complex Δ is

$$I_\Delta = (\mathbf{x}^\tau \mid \tau \notin \Delta).$$

If A is an ASM, we write Δ_A for the Stanley–Reisner complex of $\mathtt{in}\, I_A$, and I_A will mean I_{Δ_A}.

Given the Stanley-Reisner correspondence, we can consider the algebraic analogues of the deletion and link operations. Using this correspondence and definition 2.6, one can check that if $I := I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ and y is a vertex in Δ, then the ideal corresponding to the link (resp. deletion) at y can be written as $C_{y,I} + (y)$ (resp. $N_{y,I} + (y)$), where none of the generators of $C_{y,I}$ (resp. $N_{y,I}$) are divisible by y. In particular:

- $N_{y,I}$ is generated by the monomials not divisible by y.
- $C_{y,I}$ is generated by the monomials not divisible by y, as well as the quotient of the monomials divisible by y divided by y.

Definition 2.8. Let A and A' be ASMs. If A' is a submatrix of A, then we will say that A *contains* A'. Otherwise, we will say that A *avoids* A'. These definitions coincide with the usual uses of "contains" and "avoids" in the sense of pattern avoidance when A and A' are permutation matrices.

If A, B are $n \times n$ ASMs, define $A \geq B$ if $\mathrm{rk}_A(i,j) \leq \mathrm{rk}_B(i,j)$ for all $1 \leq i, j \leq n$. Restricted to permutation matrices, this is the *Bruhat order* on S_n. Let

$$\mathrm{Perm}(A) = \{w \in S_n : w \geq A \text{ and if } w \geq v \geq A \text{ for some } v \in S_n, \text{ then } v = w\}.$$

Weigandt [Wei18] showed that I_A has the irredundant prime decomposition

$$I_A = \bigcap_{w \in \mathrm{Perm}(A)} I_w.$$

The height of I_w is the Coxeter length $\ell(w)$.

Definition 2.9. Call A *equidimensional* if all elements of $\mathrm{Perm}(A)$ have the same Coxeter length.

3. ASM varieties differ from Schubert determinental varieties

First we show that it is not possible to remove an essential box and preserve Cohen–Macaulayness.

Example 3.1. *We given an example of a CM ASM A and transposition t so that tA is not CM and t "cancels a box" in the dominant region of A.*

Let $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ *and* $t = t_{1,2}$ *be the transposition of* 1 *and* 2*, in which case* $tA = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. *Then* $\mathtt{in}\, I_A = (z_{11}, z_{12}, z_{21}, z_{22}z_{31})$*, which is a complete intersection ideal and hence CM, while* $\mathtt{in}\, I_{tA} = (z_{1,1}, z_{2,1}, z_{22}z_{31}, z_{12}z_{31})$*, which is not even unmixed (hence not CM).*

Knutson and Miller [KM05] showed that matrix Schubert varieties are Cohen–Macaulay by proving that the Stanley–Reisner complex associated to their anti-digaonal initial ideals is vertex decomposable. To do so, they decompose at the vertices corresponding to the variables from largest to smallest according to the lexicographic order

$$z_{1,n} > z_{1,n-1} > \cdots > z_{1,1} > z_{2,n} > z_{2,n-1} > \cdots > z_{n,1}.$$

Note 3.2. We skip over the vertices which do not appear in the Stanley–Reisner complex $\Delta_{\mathtt{in}\, I_A}$.

Example 3.3. *Consider the following ASM and its antidiagonal initial ideal:*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad I = \mathrm{in}(I_A) = (z_{11}, z_{21}, z_{12}z_{31}, z_{31}z_{22}, z_{22}z_{13}).$$

Then the Knutson-Miller vertex decomposition method tells us to decompose at the vertex corresponding to z_{13}*. However, the deletion at the vertex corresponding to* z_{13} *is*

$$N_{z_{13},I} = (z_{11}, z_{21}, z_{12}z_{31}, z_{31}z_{22}),$$

which is not Cohen-Macaulay.

We cannot necessarily add a row and column that are all zeros except where they intersect (where we have a 1) and preserve Cohen–Macaulayness.

Example 3.4. *Let* $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $\widetilde{A} = \left(\begin{array}{cc|c|c} 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right)$. *Then A is Cohen–Macaulay but $\widetilde{A}$ is not.*

4. On one ASM containing another

Lemma 4.1. *Let $A \in ASM(n)$ and Z a generic $n \times n$ matrix. Let M be a submatrix of Z whose determinant is a Fulton generator of I_A. The antidiagonal of M must lie weakly above the antidiagonal of Z.*

Proof. Say that (p, q) lies on the k^{th} antidiagonal of Z if $p + q - 1 = k$. So, the main antidiagonal of Z has label n. Let $Z_{[i],[j]}$ be the submatrix containing the first i rows and first j columns of Z.

We claim it suffices to prove that $\text{rk}_A(i, j) \geq (i+j)-n$. If M is a square submatrix of $Z_{[i],[j]}$ whose determinant is a Fulton generator of I_A, then by definition M has size at least $\text{rk}_A(i, j) + 1 \geq (i + j + 1) - n$. Note that a diagonal of size ℓ in $Z_{[i],[j]}$ lies on the diagonal with label at most $(i + j) - (\ell - 1)$. This means that an entry on the antidiagonal of M must have value at most

$$(i + j) - (((i + j + 1) - n) - 1) = n.$$

So, the main antidiagonal of M lies weakly above the antidiagonal of Z.

Suppose $\text{rk}_A(i, j) < (i + j) - n$. Since the sum of the entries in the last $n - j$ columns of A is $n - j$ and the sum of the last $n - i$ rows of A is at most $n - i$, the sum of all entries of A is at most

$$\text{rk}_A(i, j) + (n - j) + (n - i) < n.$$

This contradicts the fact that the sum of the entries of an ASM is n. Therefore, $\text{rk}_A(i, j) \geq (i + j) - n$, which suffices for the proof. $\square$

Lemma 4.2. *Suppose A is an ASM and A' is a submatrix of A that is also an ASM. Then, Δ_A contains a subcomplex isomorphic to $\Delta_{A'}$.*

Proof. It suffices to show that $\texttt{in}\, I_A$ is contained in $\texttt{in}\, I_{A'}$ (both considered as ideals of the polynomial ring in the entries of A). Equivalently, the non-faces of Δ_A are also non-faces of $\Delta_{A'}$, which means the faces in Δ_A are also faces of $\Delta_{A'}$.

Suppose $a'_{i,j}$ is an entry in A'. Then since A is an ASM, $\text{rk}_{A'}(i, j) \leq \text{rk}_A(i, j)$. Also, note that a diagram box in the A' part of A is also a diagram box in A'. Suppose m is a minimal generator in $I_{A'}$, so that it is the antidiagonal of a minor associated to a diagram box $a'_{i,j}$. If this generator has a factor that is in A but not in A', this generator is automatically in $\texttt{in}\, I_{A'}$. Now suppose that this generator has only variables in A'. Then this generator must be contained in $\texttt{in}\, I_{A'}$ as well, since the rank of its associated diagram box in A is greater than or equal to the rank of its associated diagram box in A'. Hence, $\texttt{in}\, I_A$ is contained in $\texttt{in}\, I_{A'}$. $\square$

This next lemma is an algebraic result that we need to discuss a decomposition of an ASM A into a block-matrix in the following manner:

$$A = \left(\begin{array}{c|c} 0 & A_1 \\ \hline A_2 & 0 \end{array}\right),$$

where both A_1 and A_2 are ASMs. The result essentially states that A is Cohen-Macaulay if and only if A_1 and A_2 are both Cohen-Macaulay.

Lemma 4.3. *Let $S_1 = \mathbb{C}[x_1, \ldots, x_n]$, $S_2 = \mathbb{C}[y_1, \ldots, y_m]$, and $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Suppose that I is a proper homogeneous monomial ideal of S_1 and that J is a proper homogeneous monomial ideal of S_2. Then $R/(IR+$*

JR) is Cohen–Macaulay if and only if S_1/I is Cohen–Macaulay and S_2/J is Cohen–Macaulay.

Proof. The problem is to prove that S_1/I and S_2/J are CM if and only if $S_3 \coloneqq S_1/I \otimes_{\mathbb{C}} S_2/J$ is. It suffices to consider the depth at the homogeneous maximal ideal, and so we will abuse notation and write quotients of polynomial rings throughout this lemma to mean their localizations at their homogeneous maximal ideals.

The following is Theorem 23.3 in [Mat87]:

Lemma 4.4. *Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be Noetherian local rings, and $A \to B$ a local ring homomorphism. Let M be a finite A-module, N a finite B-module, and assume N is flat over A. Then*

$$depth_B(M \otimes_A N) = depth_A M + depth_B(N/\mathfrak{m}N).$$

We want to apply this statement in the case that $M = A = S_1/I$ and $N = B = S_3$. The inclusion $S_1/I \to S_3$ is local, since the image of the maximal ideal $(x_1, \dots, x_n)$ of S_1/I is inside the maximal ideal $(x_1, \dots, x_n, y_1, \dots, y_m)$ of S_3. S_3 is flat over S_1/I, since tensoring an S_1/I-module with $S_3 = S_1/I \otimes_{\mathbb{C}} S_2/J$ (over S_1/I) is tantamount to tensoring it with S_2/J over $\mathbb{C}$, and S_2/J is flat over $\mathbb{C}$. So the hypotheses are satisfied, and we end up with

$$\mathrm{depth}_B B = \mathrm{depth}_B(A \otimes_A B) = \mathrm{depth}_A A + \mathrm{depth}_B(B/\mathfrak{m}B).$$

Now, since $B = R/(IR + JR)$, $\mathfrak{m}R$ contains IR, and $R/\mathfrak{m}R \cong S_2$, it follows that $B/\mathfrak{m}B \cong S_2/J$. Next, because $\mathfrak{m}$ kills S_2/J, $b_1, \dots, b_i$ is a regular sequence for S_2/J as a B-module if and only if their images in $B/\mathfrak{m}B$ is a regular sequence for S_2/J as a $B/\mathfrak{m}B = S_2/J$-module. So what we get is

$$\mathrm{depth}(S_3) = \mathrm{depth}(S_1/I) + \mathrm{depth}(S_2/J).$$

It remains to show the equality

$$\dim(S_3) = \dim(S_1/I) + \dim(S_2/J).$$

Let $d_1 = \dim(S_1/I), d_2 = \dim(S_2/J), d_3 = \dim(S_3)$. By Noether normalization we may find sets of algebraically independent elements $\{\alpha_1, \dots, \alpha_{d_1}\}$ and $\{\beta_1, \dots, \beta_{d_2}\}$ such that S_1/I is finite over $\mathbb{C}[\alpha_1, \dots, \alpha_{d_1}]$ and S_2/J is finite over $\mathbb{C}[\beta_1, \dots, \beta_{d_2}]$. Then their tensor product S_3 is finite over $\mathbb{C}[\alpha_1, \dots, \alpha_{d_1}] \otimes_{\mathbb{C}} \mathbb{C}[\beta_1, \dots, \beta_{d_2}] \cong \mathbb{C}[\alpha_1, \dots, \alpha_{d_1}, \beta_1, \dots, \beta_{d_2}]$ (where the $\alpha_1, \dots, \alpha_{d_1}$, $\beta_1, \dots, \beta_{d_2}$ are algebraically independent because the α's only involve x-variables, and the β's only involve y-variables). Again by Noether normalization, this shows that $d_1 + d_2 = d_3$.

The above two displayed equations show that if S_1/I and S_2/J are Cohen-Macaulay, then so is S_3. For the reverse direction, we use the two equations along with the inequality $\mathrm{depth}(R) \leq \dim(R)$, valid for general (local) rings R. □

The following lemma discusses the restrictions on the additional entries when an ASM A' is embedded into an ASM A as a submatrix.

Lemma 4.5. *Suppose a dimension n ASM A has an ASM submatrix A', and that the part of A not in A' are rows $R = \{r_1, \dots, r_k\}$ and columns $C = \{c_1, \dots, c_k\}$, ordered. Then the following must hold:*

(1) *For all r_i, $A_{r_i,c} = 0$ if $c < c_1$ or $c > c_k$, and for all c_i, $A_{r,c_i} = 0$ if $r < r_1$ or $r > r_k$.*

(2) $\sum_{r_i \in R, c_j \in C} A_{r_i,c_j} = k$. *This implies there are at least k pairs (r_i, c_j) such that $A_{r_i,c_j} = 1$.*

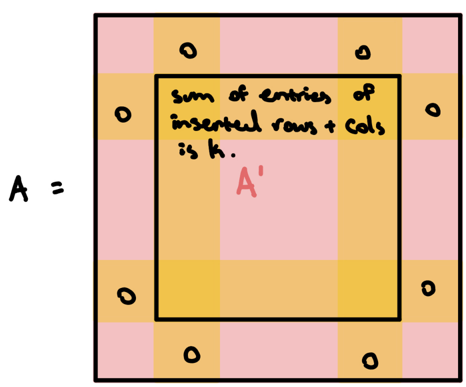
Proof. For the first property, we will show this for the case that $c < c_i$. The argument is exactly the same for the other cases. For the sake of contradiction, suppose $A_{r_i,c} \neq 0$. If $A_{r_i,c} = -1$, then there must be some $c' < c$ such that $A_{r_i,c'} = 1$. If $A_{r_i,c} = 1$, noting that in A', there must be some row r such that $A_{r,c} = 1$, so there must be some r_j between r and r_i such that $A_{r_j,c} = -1$. Then, from this new entry that we know must be nonzero, we can inductively apply this argument, and we see that there must be a -1 in column 1 of the matrix, a contradiction to the fact that A is an ASM.

For the second property, first note that for c not in C, $\sum_{r_i \in R} A_{r_i,c} = 0$, and that for r not in R, $\sum_{c_i \in C} A_{r,c_i} = 0$, since each row and column must sum to 1, and this is already true of A'. Then, consider the rank of $A_{n,n}$. Note that it must be true that $\mathrm{rk}(A_{n,n}) = n = \mathrm{rk}(A'_{n-k,n-k}) + k$, since A' and A are both ASMs. Then it follows that

$$k = \sum_{r \in R \text{ or } c \in C} A_{r,c} = \sum_{r_i \in R \text{ and } c \notin C} A_{r_i,c} + \sum_{r \notin R \text{ and } c_i \in C} A_{r,c_i} + \sum_{r_i \in R \text{ and } c_i \in C} A_{r_u,c_i}$$
$$= \sum_{r_i \in R \text{ and } c_j \in C} A_{r_i,c_j}.$$

Hence, the intersections of the inserted rows and columns must have at least k nonzero entries. □

This lemma essentially states that the additional rows and columns must be zero outside the "intersection box" and sum to k inside the "intersection box".



This lemma also directly implies that if A' embeds in A as a block in the north-west corner, i.e. $A = \left(\begin{array}{c|c} A' & * \\ \hline * & * \end{array}\right)$, then the rest of so the matrix A must only be nonzero in the block in the south-east corner, i. e. $A = \left(\begin{array}{c|c} A' & 0 \\ \hline 0 & * \end{array}\right)$.

In order to examine potential height unmixedness of an ASM, we will find it useful to have combinatorial properties and algorithms that allow us to recognize the minimal primes corresponding to an ideal of an ASM. The followimg lemma gives such a property.

Lemma 4.6. *Let I be a monomial ideal of degree 1 in each variable, and let S be a subset of the variables that appear in I. Let I_S be the ideal generated by the the variables in S. Then I_S is a minimal prime corresponding to I if and only if the following hold.*

(i) *At least one variable from each monomial in I is contained in S.*
(ii) *For every element of S, there exists a monomial in I which contains that variable but no other variables in S.*

Proof. Consider Δ_I, the Stanley-Reisner ideal of I. We can view S as a subset of the vertices in Δ_I. Then I_S is a minimal prime corresponding to I if and only if S^c, the complement of S,is a facet of Δ_I. We will show that this is the case.

First, (i) implies that for any monomial in I, not all of the variables in it are contained in S^c. Thus S^c is a face of Δ_I. Then (ii) implies that if we were to add any other vertex from S into S^c, we would no longer have a face. Thus S^c is a maximal face. □

Using Lemma 4.6, we can show that various families of ASMs are not equidimensional (and therefore not Cohen-Macaulay) by finding two minimal primes which we demonstrate to be of different heights. We give such a class of ASMs now.

Proposition 4.7. *Let A be an ASM which satisfies the following properties:*

(1) *A contains the block $B = \begin{array}{cc} & \boxed{0} \\ \boxed{1} & \boxed{-1} \end{array}$, with the 1 falling in row r and column c.*
(2) *All entries of A northwest of the 1 in B are zeros.*
(3) *All essential boxes of* $\operatorname{rk} A$ *which don't correspond to a box from B are either rank 0 or rank at least $r-1$.*
(4) *A has no essential boxes in column c.*
Then A is not equidimensional.

Proof. To show that A is not equidimensional, we will construct two different minimal primes of $\texttt{in}\, I_A$ and show that they have different height.

Let $I_{z_{i,j}}$ be the ideal generated by monomials of $\texttt{in}\, I$ all of whose variables come at or before $z_{i,j}$ in lexicographic order. We begin by finding two different minimal primes of $I_r = I_{z_{r,n}}$ which have different height. We then induct on our vertex to extend these to minimal primes of $I_{z_{i,j}}$ which at each step continue to have different height.

Draw the grid of points representing the variables $z_{i,j}$ in the generic matrix for A. For each monomial generator in our antidiagonal ideal, we can then add a line/curve connected the set of points of variables contained in that monomial. Figure 3 shows such a drawing for a possible I_r on the variables $z_{1,1}, \dots, z_{r,n}$.

Given such a drawing for I_r, we draw an orange circle around the top vertex of each monomial generator of I_r which has degree greater than 1. Call this set of vertices O_r. Now we draw a yellow circle around the bottom vertex of each monomial generator of I_r which has degree greater than 1, and call this set of vertices Y_r.

Since all monomials in I_r contain some variable in row r, we will have $Y_r = \{z_{r,c}, \dots, z_{r,c+i}\}$ for some $i \geq 0$. Then O_r will consist of the set of variables

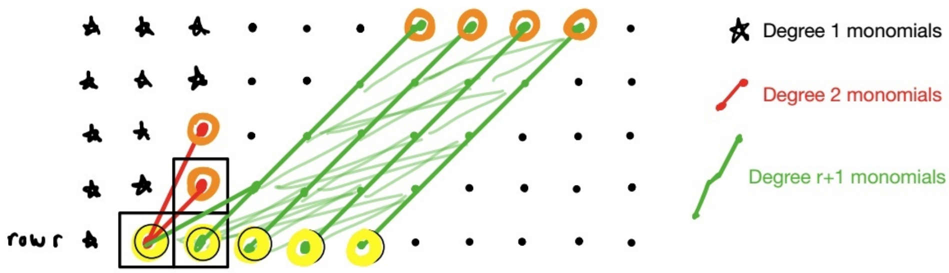


FIGURE 3. The grid of variables z_i, j for $i \leq r$. Monomial generators of I_r are drawn in with lines and stars. The variables in O_r are circled in orange at the top of each monomial, and those in Y_r are circled in yellow at the bottom of each monomial.

$\{z_{1,c+r}, \ldots, z_{1,c+i+r-1}\}$ in row 1 which correspond to the tops of degree r monomials, as well as at least two more vertices in column $c+1$ which correspond to the tops of our rank 2 monomials. The set O_r will always have more vertices than the set Y_r.

Both Y_r and O_r are minimal primes of I_r since every generator of I_r (with degree greater than one) contains an orange circle at the top and a yellow circle at the bottom, and further that generator has only one orange and one yellow circle. Thus I_r is not height unmixed.

Now we will induct on our vertex $z_{i,j}$ by adding orange (yellow) circles to O_r (Y_r) to extend them to sets $O_{z_{i,j}}, Y_{z_{i,j}}$ which generate minimal primes for $I_{z_{i,j}}$, while maintaining at each step that $|O_{z_{i,j}}| > |Y_{z_{i,j}}|$. Our rule for adding circles is as follows:

Let $Y_{z_{r,n}} = Y_r$ and $O_{Z_{r,n}} = O_r$. Given vertices z and z' such that z' directly follows z in lexicographic order, let

$$Y_{z'} = \begin{cases} Y_z \cup \{z'\} & \text{if } \exists \text{ monomial } m \in I_{z'} \text{ s.t. } z'|m, \text{ but m has no circles in } Y_z \\ Y_z & \text{if } \forall \text{ monomials } m \in I_{z'} \text{ s.t. } z'|m, \text{some variable in } m \text{ is in } Y_z. \end{cases}$$

and similarly for O'_z, replacing each instance of Y with O. We claim that each vertex $z \geq z_{r,n}$ is in Y_z exactly when it is in O_z. This is true for our base case, $I_{z_{r,n}}$. We will now assume the property holds for I_z and show that it is true for $I_{z'}$ where z' directly follows z in lexicographic order.

For a monomial $m \in I_{z'}$ for which $z \mid m$, let the *base* of m be the variables in m which fall below row r, and the *top* of m be the variables which fall in row r or above. For a given base b with bottom variable z, consider all of the monomials in $m \in I_{z'}$ such that $b \mid m$. By inductive hypothesis, b contains an orange circle exactly when it contains a yellow circle. In this case, all monomials with base b already contain both an orange and a yellow circle.

If b does not contain an orange/yellow circle, we consider what possible tops t exist for monomials in $I_{z'}$ with base b. Let d be the lowest possible degree of t. Then $t = z_{r,k} z_{r-1,k+1} \cdots z_{r-d+1,k+d}$ is a valid top for some $k \geq c$. If $d = r$, then t contains either both a yellow and an orange circle or t contains neither.

Now suppose $d < r$. If we can choose such a $k > c$, then bt contains no orange circles, so $z' \in O_{z'}$. Further $t' = z_{r-1,k}z_{r-2,k+1}\cdots z_{r-d,k+d}$ (all of the vertices directly above each vertex in t) is also a valid top, and bt' contains no yellow circles, so $z' \in Y_{z'}$. This case is illustrated in Figure 4. If instead the only possible choice of top has $k = c$ (the leftmost block falls in column c), then by property 4 of Proposition 4.7, t has at least degree two, and thus contains both a yellow and an orange circle.

Thus after going through all bases and possible monomials containing each base, we find $z' \in O_{z'}$ exactly when $z' \in Yz'$.

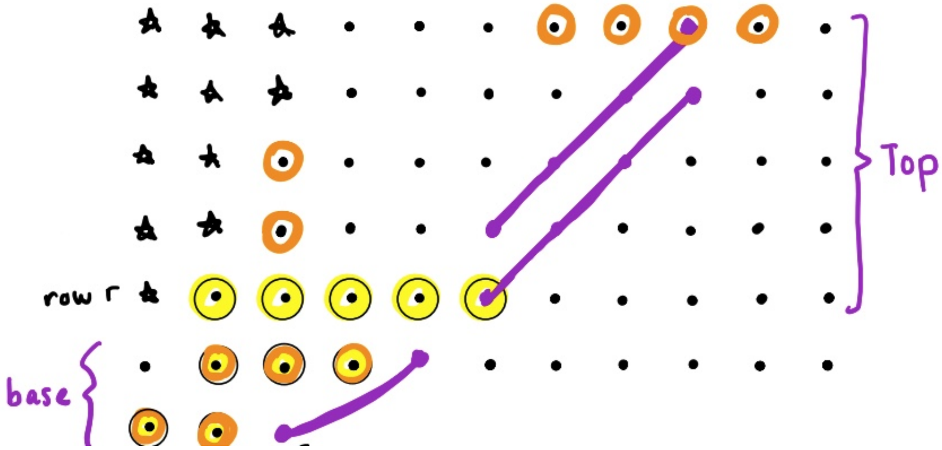


FIGURE 4. The induction process on the ideal from figure 3. Note that all vertices below row r have both a yellow or orange circle. One base is drawn in, as well as possible tops t and t'.

This gives that $O_{z_{n,n}}$ and $Y_{z_{n,n}}$ contain the exact same elements from row after r but a different number of elements in rows r and above. Thus we have found two minimal primes of $\mathtt{in}\, I_A$ which have different heights.

□

The proof that this particular family of ASMs is not equidimensional hinges on the containment of degree two monomials (as drawn in red in Figure 3) which form a height not unmixed ideal, and fall in a particular way with regards to the other monomials in I_A, allowing us to inductively build up our minimal primes of different heights. One could likely use this proof method to prove unmixedness of other families of ASMs that have different monomials in the first r rows that can be shown to be not height unmixed and which further fall in ways compatible with induction on later vertices.

5. Results on unmixedness and Cohen–Macaulayness of ASMs

In this section, we consider the operation we called "adding a one" in the introduction. We start with an $n \times n$ matrix A and build an $(n+1) \times (n+1)$ matrix $\widetilde{A} = \widetilde{A}(i, j)$ according to the following rules:

(1) The (i, j) entry of $\widetilde{A}$ is 1, and the entries in row i and column j are zero otherwise.

(2) The entries in rows other than i and columns other than j form the matrix A.

Visually, $\widetilde{A} = \widetilde{A}(i,j)$ is the matrix

$$\widetilde{A} = \left(\begin{array}{c|c|c} & 0 & \\ A_1 & \vdots & A_2 \\ & 0 & \\ \hline 0\dots0 & 1 & 0\dots0 \\ \hline & 0 & \\ A_3 & \vdots & A_4 \\ & 0 & \end{array} \right), \tag{1}$$

First we demonstrate that by taking a series of links in the top row, if $\widetilde{A}$ is Cohen-Macaulay then "removing a one" lying in the top row preserves Cohen-Macaulayness.

Proposition 5.1. *Let $A = [A_1|A_2]$ be an ASM. Let $\widetilde{A} = \widetilde{A}(i,1)$ for any $i = 1,\dots,n$. If $\widetilde{A}$ is Cohen-Macaulay then A is as well.*

Proof. Let

$$M_A = \begin{pmatrix} x_{1,1}\dots x_{1,k} & z_{1,1}\cdots z_{1,n-k} \\ \vdots & \vdots \\ x_{n,1}\cdots x_{n,k} & z_{n,1}\cdots z_{n,n-k} \end{pmatrix}$$

.

and let

$$M_{\widetilde{A}} = \left(\begin{array}{c|c|cc} x_{0,1}\dots x_{0,k} & z_{0,1} & z_{0,2}\cdots z_{0,n-k} & z_{0,0} \\ \hline x_{1,1}\dots x_{1,k} & z_{1,1} & z_{1,2}\cdots z_{1,n-k} & z_{1,0} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n,1}\cdots x_{n,k} & z_{n,1} & z_{n,2}\cdots z_{n,n-k} & z_{n,0} \end{array} \right)$$

.

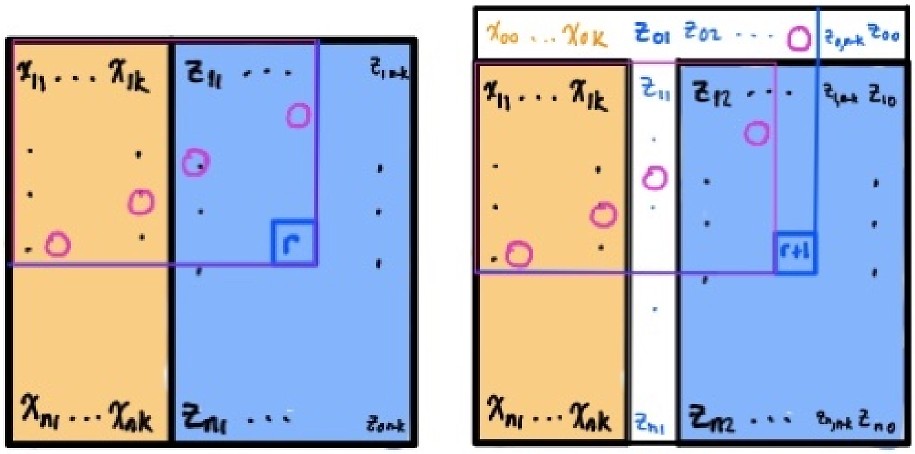
The element $x_{0,1}$ of $M_{\widetilde{A}}$ is in the spot of a corresponding essential box of $\widetilde{A}$ which has rank 0. Thus $x_{0,1},\cdots,x_{0,k}$ all appear in $\mathtt{in}\, I_{\widetilde{A}}$.

For $i,j > 0$, the element $x_{i,j}$ of $M_{\widetilde{A}}$ corresponds to an essential box of $\widetilde{A}$ exactly when $x_{i,j}$ of M corresponds to an essential box of A. Further, these essential boxes have the same rank. Thus any anti-diagonal products coming from such an essential box are going to be the same for $\mathtt{in}\, I_A$ and $\mathtt{in}\, I_{\widetilde{A}}$. (Note that in theory $\mathtt{in}\, I_{\widetilde{A}}$ could also have anti-diagonal products that include one of $x_{0,1},\cdots,x_{0,k}$. However these are already in $\mathtt{in}\, I_{\widetilde{A}}$, so writing in as a generator an anti-diagonal product containing one of them would be redundant.

There will be no essential boxes in the added column or in the added row to the right of the one. For all essential boxes in the A_2 region, we can do a process similar to what we did for when we added the one in the top left.

□

Now we consider the converse direction, where we start with a matrix $\widetilde{A}$ of the form in Equation (1), and consider when the matrix A obtained from $\widetilde{A}$ by ignoring row i and column j is Cohen–Macaulay (resp. equidimensional) provided $\widetilde{A}$ is Cohen–Macaulay (resp. equidimensional).



(A) The matrix M_A, with an essential box of rank r squared, and antidiagonal elements for a corresponding $(r+1) \times (r+1)$ minor circled.

(B) The matrix $M_{\widetilde{A}}$ with the same essential box, now or rank r+1 circled. Anti-diagonal elements circled have the same labels as M_A with one additional variable from the top added row.

FIGURE 5. Anti-diagonals for an essential box that falls in A_2.

Proposition 5.2. *Let A be an $n \times n$ ASM. Then the following are equivalent:*

(1) *A is Cohen-Macaulay.*
(2) *$\widetilde{A}(n+1, n+1)$ is Cohen–Macaulay.*
(3) *$\widetilde{A}(1, n+1)$ is Cohen–Macaulay.*
(4) *$\widetilde{A}(1, n)$ is Cohen–Macaulay.*
(5) *$\widetilde{A}(2, n+1)$ is Cohen–Macaulay, provided the $(1, n)$ entry of A is 1.*

Conditions (1) through (4) of 5.2 are illustrated in figure 6.

Proof. From Theorem 2.5, we know that I_B is Cohen-Macaulay if and only if $\mathtt{in}\, I_B$ is Cohen-Macaulay for any ASM B. So, it suffices to show that $\mathtt{in}\, I_{\widetilde{A}}$ is Cohen-Macaulay if and only if $\mathtt{in}\, I_{\widetilde{A}}$ is Cohen-Macaulay. We split into cases according to which decomposition we are in.

(1) There are no essential boxes in the last row and column of $\widetilde{A}$. The essential boxes of $\widetilde{A}$ are then exactly the same as those of A. Thus $\mathtt{in}\, I_{\widetilde{A}} = \mathtt{in}\, I_A$.
(2) The $(1, n)$ entry is an essential box with value 0. All other essential boxes of A are essential boxes of A with the same value. Moreover, the antidiagonal terms containing $z_{1,1}, \ldots, z_{1,n}$ are absorbed in the ideal $(z_{1,1}, z_{1,2}, \ldots, z_{1,n})$. So,
$$\mathtt{in}\, I_{\widetilde{A}} = (z_{1,1}, z_{1,2}, \ldots, z_{1,n}) + \mathtt{in}\, I_A.$$
The variables $z_{1,1}, \ldots, z_{1,n}$ are independent of $\mathtt{in}\, I_A$, so $\mathtt{in}\, I_{\widetilde{A}}$ is Cohen-Macaulay exactly when $\mathtt{in}\, I_A$ is.
(3) The $(1, n-1)$ entry is an essential box with value 0. Note that A does not have essential boxes in the last column because it will be covered by a one

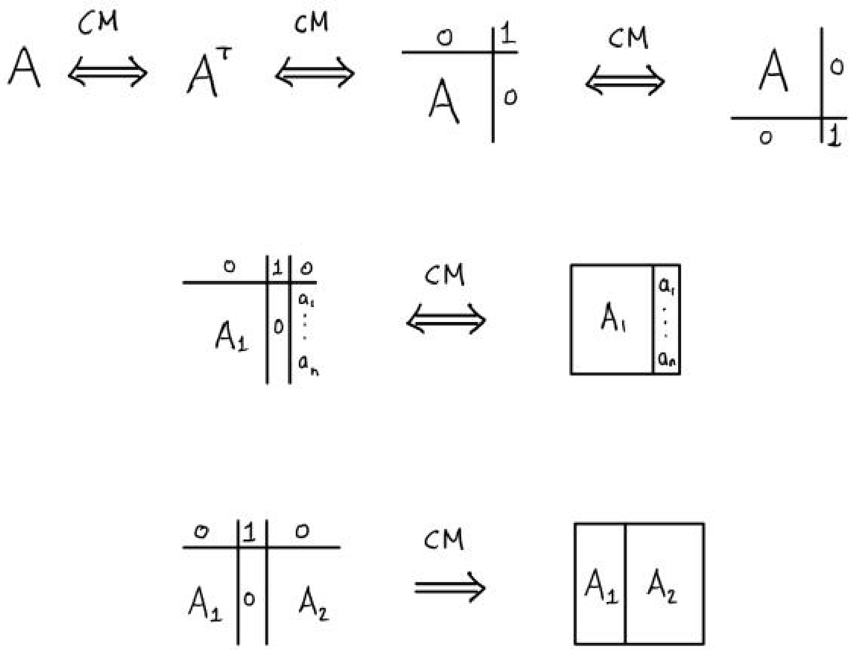


FIGURE 6. Some cases where "adding (or removing) a one" preserves Cohen-Macaulayness.

somewhere to the left. So we get a similar decomposition

$$\texttt{in}\, I_{\widetilde{A}} = (z_{1,1}, z_{1,2}, \ldots, z_{1,n-1}) + \texttt{in}\, I_A,$$

which implies that $\texttt{in}\, I_{\widetilde{A}}$ is Cohen-Macaulay if and only if $\texttt{in}\, I_A$ is.

(4) Since A has a 1 in the top right corner, we note that there cannot be an essential box in the last column of A (which is the second to last column of $\widetilde{A}$): an essential box can only arise from a (-1) in entry (i, n) for $i \geq 3$, which implies that there is a 1 in entry $(i, n+1)$, but this contradicts the fact that the last column of $\widetilde{A}$ sums to 1. Thus, we are in case 2, so $\texttt{in}\, I_{\widetilde{A}}$ is Cohen-Macaulay if and only if $\texttt{in}\, I_A$ is.

□

Theorem 5.3. *Let A be an $n \times n$ ASM and $\widetilde{A} = \widetilde{A}(1, j)$ for $j = 1, 2, \ldots, n+1$. If A is unmixed, then A' is unmixed.*

To prove the theorem, we prove a bijection between $\text{Perm}(A)$ and $\text{Perm}(\widetilde{A})$.

Lemma 5.4. *Let A be an $n \times n$ ASM and $\widetilde{A} = \widetilde{A}(1, j)$ for $j = 1, 2, \ldots, n+1$. Then there is a bijection between $\text{Perm}(A)$ and $\text{Perm}(\widetilde{A})$: for $w \in \text{Perm}(A)$ written in one-line notation $(w_1, \ldots, w_n)$, the corresponding $w' \in \text{Perm}(\widetilde{A})$ is $(j, w'_2, \ldots, w'_{n+1})$, where $w'_k = w_k + 1$ if $w_k \geq j$ and $w'_k = w_k$ otherwise.*

Proof. Let $w \in S_n$ be a permutation, and let w' be the corresponding permutation under the map described in the proposition, i.e. if $w = (w_1, \dots, w_n)$, then $w' = (j, w'_2, \dots, w'_{n+1})$, where $w_k = w_k + 1$ if $w_k \geq j$ and $w'_k = w_k$ otherwise. This corresponds to inserting a column in the j^{th} position of the permutation matrix of w. For an ASM B, let $B(k)$ be the column index of the leftmost 1 in row k.

Suppose $w \in \mathrm{Perm}(A)$. First we show that $w' \in \mathrm{Perm}(\widetilde{A})$. If we insert a column in an ASM B after column j to get an ASM B', then comparing the rank matrices, we see that

$$\begin{cases} \mathrm{rk}_{B'}(1,b) = 1 & \text{for } b \geq j, \\ \mathrm{rk}_{B'}(1,b) = 0 & \text{for } b < j, \\ \mathrm{rk}_{B'}(a+1,b) = \mathrm{rk}_B(a,b) & \text{for } a = 1, \dots, n \text{ and } b < j, \\ \mathrm{rk}_{B'}(a+1,j) = \mathrm{rk}_B(a,j) + 1 & \text{for } a = 1, \dots, n, \\ \mathrm{rk}_{B'}(a+1,b+1) = \mathrm{rk}_B(a,b) + 1 & \text{for } a = 1, \dots, n \text{ and } b \geq j. \end{cases}$$

Restricting to the pairs $(B, B') = (w, w')$ and $(B, B') = (A, \widetilde{A})$, we see that

$$w \geq A \text{ if and only if } w' \geq \widetilde{A}. \tag{2}$$

If there exists a $\tilde{v}$ such that $w' > \tilde{v} \geq A$, then

$$\mathrm{rk}_A(a,b) \geq \mathrm{rk}_{\tilde{v}}(a,b) \geq \mathrm{rk}_{w'}(a,b)$$

for all $1 \leq a, b \leq n+1$. Taking $a = 1$, we see that the first row $\tilde{v}$ is the same as the first row of $\widetilde{A}$ (which is the same as the first row of w'). This means that $\tilde{v} = v'$ for some $v \in \mathrm{Perm}(A)$. Then $w' > v' \geq \widetilde{A}$ implies $w > v \geq A$, contradicting $w \in \mathrm{Perm}(A)$.

It suffices to show that for every $\tilde{v} \in S_n$ such that $\tilde{v} \in \mathrm{Perm}(\widetilde{A})$, we have $\tilde{v} = v'$. To see this, first we note that $\tilde{v} \geq \widetilde{A}$ implies $v \geq A$ from (2). If there exists $u \neq v$ for which $v \geq u \geq A$, this implies $v' \geq u' \geq \widetilde{A}$, contradicting $\tilde{v} = v' \in \mathrm{Perm}(A)$. So, $v \in \mathrm{Perm}(A)$, which completes the correspondence between $\mathrm{Perm}(A)$ and $\mathrm{Perm}(\widetilde{A})$.

If there is no $v \in S_n$ such that $\tilde{v} = v'$, since $\tilde{v} \geq \widetilde{A}$, we can suppose for sake of contradiction that $\tilde{v}(1) > j$. Let k be the smallest row for which $j \leq \tilde{v}(k) < \tilde{v}(1)$. Let R be the grid rectangle with the top right vertex at the entry $(1, \tilde{v}(1))$ and bottom left rectangle $(k, \tilde{v}(k))$. Let the black dots inside R represent the ones in the permutation matrix of $\tilde{v}$, see Figure 7.

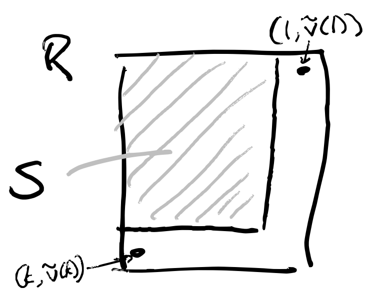


FIGURE 7

Let $\tilde{u} \in S_n$ be the permutation obtained from $\tilde{v}$ by taking $\tilde{u}(1) = \tilde{v}(k)$ and $\tilde{u}(k) = \tilde{v}(1)$ and $\tilde{u}(j) = \tilde{v}(j)$ otherwise. This corresponds to replacing black dots on the antidiagonal of R (shown in the figure) with black dots on the main diagonal. Since $v \leq \widetilde{A}$ and $\tilde{v}(1) > j$, $\mathrm{rk}_{\tilde{v}}$ is equal to 0 in the gray region S of R shown below obtained by removing the bottom row and last column of R. Since $\widetilde{A}(1) = j$, rk_A is equal to 1 in S. Then, the rank matrix $\mathrm{rk}_{\tilde{u}}$ increases by 1 in S and is otherwise equal to $\mathrm{rk}_{\tilde{v}}$. This shows that $\tilde{u}$ satisfies $A \leq \tilde{u} \leq \tilde{v}$, contradicting minimality of v. □

Proof of Theorem 5.3. Since I_A is unmixed, $\ell(w) = \ell(v)$ for all $w, v \in \mathrm{Perm}(A)$. Let $f : \mathrm{Perm}(A) \to \mathrm{Perm}(\widetilde{A})$ be the bijection from Lemma 5.4. It follows from the definition of f that $f(w) = w + c$ for some constant c for some c independent of any $w \in \mathrm{Perm}(A)$. This implies that all permutations $f(w) \in \mathrm{Perm}(A)$ have the same Coxeter length. This suffices for the proof. □

6. Further questions

Related to Proposition 5.2 are the following questions.

Question 6.1. *Let A be an ASM and $\widetilde{A} = A(i,j)$ for some $1 \leq i,j \leq n$. Is there an example of an ASM A such that A is not Cohen–Macaulay but $\widetilde{A}$ is Cohen–Macaulay?*

Question 6.2. *If A is a Cohen–Macaulay ASM, is $\widetilde{A} = A(1,1)$ Cohen–Macaulay?*

Question 6.3. *More generally, is the converse to Proposition 5.1 true?*

7. Data summary

For all ASMs up through ASM(6), whenever A is not CM and $\widetilde{A}$ contains A, $\widetilde{A}$ is also not CM.

For all ASMs up through ASM(5), whenever A is Cohen–Macaulay, Δ_A is vertex decomposable.

The following is a table of counts of Cohen-Macaulay and non-Cohen-Macaulay ASMs. Note that before ASM(4), all ASMs are Cohen-Macaukay, so we don't include them.

dimension	# CM ASMs	# not CM ASMs
4	39	3
5	328	101
6	4028	3408

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